Performance Assessment of Cascade Control Loops

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The minimum variance control law for a cascade control system is derived for two cases: the primary process has stable and unstable inverses. The expression for the primary output variable under the minimum variance cascade control is shown to be feedback-invariant. An estimate of the minimum achievable variance in a cascade control loop can be obtained via multivariate time-series modeling of the primary and secondary measurements collected under normal operation. The utility of the proposed performance assessment method is demonstrated using simulated and industrial data.

Introduction

The use of minimum achievable variance as a performance benchmark for control-loop performance assessment has been extensively studied during the last decade. Methodologies for the estimation of the best achievable performance bound in terms of output variance have been established for SISO control systems (Harris, 1989; Desborough and Harris, 1992; Stanfelj et al., 1993; Lynch and Dumont, 1996; Kozub, 1997) and MIMO control systems (Harris et al., 1996; Huang et al., 1997). Extension of the methodology to processes with unstable inverses has also been made (Tyler and Morari, 1995; Harris et al., 1996; Huang et al., 1997). Two comprehensive review articles have been recently published in this area (Qin, 1998; Harris et al., 1999).

Using minimum variance control in performance assessment is very attractive, because its estimation requires minimal knowledge of the process (only the process time delay). On the other hand, performance assessment with the minimum variance performance bound has limited utility because minimum variance control is rarely used in industrial applications due to excessive control actions and lack of robustness. Nevertheless, there are some cases where the condition of minimum variance control is indeed achieved (Qin, 1998). The minimum variance performance benchmark can thus serve as a first-level performance assessment measure; a reduced number of control loops that indicate poor performance relative to the minimum variance performance bound can be further diagnosed by higher-level performance assessment tools.

A number of performance indices have been introduced in the literature (Devries and Wu, 1978; Desborough and Harris, 1992; Stanfelj et al., 1993; Kozub and García, 1993) to provide basic information on the extent of departure of current control performance from minimum variance control. When the current control performance is found to be poor relative to minimum variance control, Kozub and García (1993) and Huang and Shah (1998) suggested using a desired closed-loop dynamic impulse response of the output to determine if the current output response characteristics are acceptable. Desborough and Harris (1992), Harris et al. (1999), and Thornhill et al. (1999) suggested using an extended horizon performance index, especially when the delay structure of the process is poorly known. Ko and Edgar (1998) proposed using the PI-achievable lower bound as a performance measure in assessing PI controller performance.

This article presents a comprehensive methodology for the performance assessment of a cascade control loop using minimum variance principles. Cascade control is one of the routinely used control strategies in the chemical industries, especially when disturbances occur in the manipulated variable or when the actuator exhibits nonlinear behavior. Despite its importance, however, performance assessment of cascade control systems has not been discussed in the literature.

Closed-loop performance can be greatly enhanced (vs. standard feedback) by employing cascade control via a secondary measurement and a secondary feedback controller. Hence, the minimum achievable variance with cascade control is generally lower than that from single-loop feedback control and can provide useful information on potential performance improvement in comparison with single loop feedback control.

The minimum variance cascade control law is derived for minimum-phase processes and a proper expression for the primary output under the minimum variance cascade control

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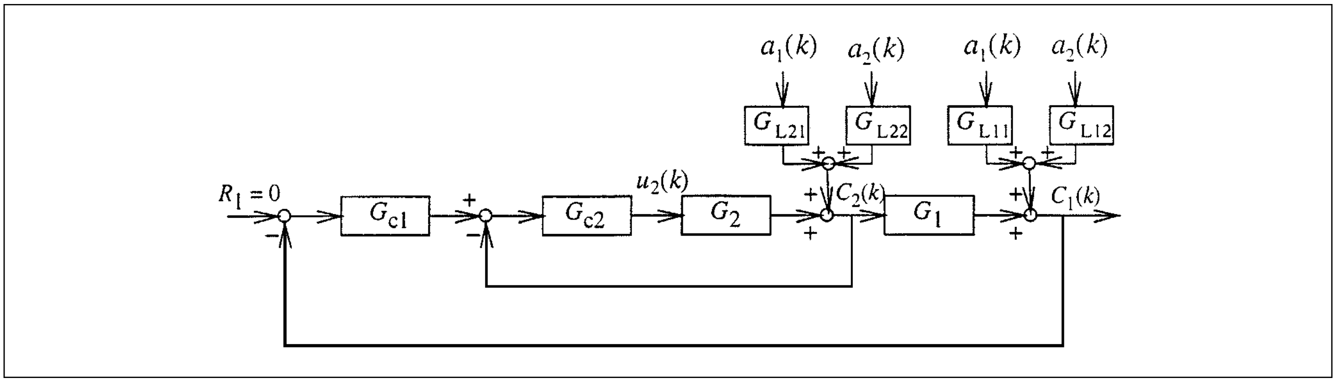

Figure 1. Cascade control system.

is obtained. A method to obtain an estimate of minimum achievable variance from routine process operating data and the knowledge of the process time delays is also discussed. The performance assessment algorithm is extended to processes having unstable inverses. The proposed performance assessment method is illustrated using a simulated and an industrial data set, followed by conclusions.

Minimum Variance Cascade Control

Derivation of minimum variance cascade control

In this section, a control law for the cascade control loop is derived that achieves minimum variance in the primary output when stochastic load disturbances occur in both the primary and the secondary loops.

Figure 1 shows a discretized cascade control system with unity feedback. Subscript 1 in this figure refers to the primary control loop, while subscript 2 refers to the secondary control loop. The disturbances that represent the net effect of noise and unmeasured disturbances in the primary loop can equivalently be added at the output of the primary loop

$$C_1(k) = G_1 C_2(k) + G_{L11} a_1(k) + G_{L12} a_2(k) \qquad (1)$$

where $C_1(k)$ and $C_2(k)$ are the process outputs of the primary and the secondary loop at discrete sampling time k. $C_1(k)$ is the deviation variable from its set point, and $C_2(k)$ is the deviation of the secondary output from its steady-state value which is required to keep the primary output at its set point. $G_1(q^{-1}) \equiv G_1^* q^{-d_1}$ is the process transfer function in the primary loop with time delay equal to d_1, and G_1^* is the primary process model without any time delay. In this section, we assume $G_1(q^{-1})$ has a stable inverse, that is, all zeros of $G_1(q^{-1})$ lie inside the unit circle. The disturbance filters $G_{L11}(q^{-1})$ and $G_{L12}(q^{-1})$ are assumed to be a rational function of the backward shift operator q^{-1}, and they are driven by zero-mean white noise sequences $\{a_1(k)\}$ and $\{a_2(k)\}$, respectively. Such a disturbance representation can handle nonstationary random processes, as well as stationary ones (MacGregor et al., 1984; Box et al., 1994).

Similary, for the secondary loop, we have

$$C_2(k) = G_2 u_2(k) + G_{L21} a_1(k) + G_{L22} a_2(k) \qquad (2)$$

where $u_2(k)$ is the manipulated variable in the secondary control loop and $G_2(q^{-1}) \equiv G_2^* q^{-d_2}$ is the process transfer function in the secondary loop with time delay equal to d_2, and G_2^* is a secondary process model without any time delay. $G_2(q^{-1})$ is also assumed to be minimum-phase. The combined effect of all unmeasured disturbances to the secondary output is represented as a superposition of disturbance filters $G_{L21}(q^{-1})$ and $G_{L22}(q^{-1})$ driven by zero-mean white noise sequences $\{a_1(k)\}$ and $\{a_2(k)\}$, respectively.

Theorem 1

(i) The minimum variance control algorithm for the cascade system 1—2 is given by

- *Primary Controller*

$$G_{c1} = \frac{G_1^*(Q_{22}R_{21} - Q_{21}R_{22}) + (R_{11} + T_1)G_{L22} - (R_{12} + T_2)G_{L21}}{\left(Q_{11} + S_1 q^{-d_1}\right)(R_{12} + T_2 + G_1^* R_{22}) - \left(Q_{12} + S_2 q^{-d_1}\right)(R_{11} + T_1 + G_1^* R_{21})} \qquad (3)$$

- *Secondary Controller*

$$G_{c2} = \frac{\left(Q_{11} + S_1 q^{-d_1}\right)(R_{12} + T_2 + G_1^* R_{22}) - \left(Q_{12} + S_2 q^{-d_1}\right)(R_{11} + T_1 + G_1^* R_{21})}{G_2^*\left[G_{L11}S_2 - G_{L12}S_1 + (R_{11}Q_{12} - R_{12}Q_{11})q^{-d_2}\right]} \qquad (4)$$

where Q_{11} and Q_{12} are polynomials in q^{-1} of order d_1+d_2-1, and Q_{21}, Q_{22}, S_1, and S_2 are polynomials in q^{-1} of order d_2-1, and R_{ij} and T_i $(i,\ j=1,2)$ are proper transfer functions that satisfy the following Diophantine identities

$$G_{L11}=Q_{11}+R_{11}q^{-d_1-d_2} \tag{5}$$

$$G_{L12}=Q_{12}+R_{12}q^{-d_1-d_2} \tag{6}$$

$$G_{L21}=Q_{21}+R_{21}q^{-d_2} \tag{7}$$

$$G_{L22}=Q_{22}+R_{22}q^{-d_2} \tag{8}$$

$$G_1^*Q_{21}=S_1+T_1q^{-d_2} \tag{9}$$

$$G_1^*Q_{22}=S_2+T_2q^{-d_2} \tag{10}$$

(ii) The primary output $C_1(k)$ under this optimal control algorithm is a moving average process of order d_1+d_2-1

$$C_1(k)=\left[Q_{11}(q^{-1})+S_1(q^{-1})q^{-d_1}\right]a_1(k)+\left[Q_{12}(q^{-1})+S_2(q^{-1})q^{-d_1}\right]a_2(k) \tag{11}$$

and the minimum variance of $C_1(k)$ is

$$\left(\sigma^2_{C_1}\right)_{MV}=\mathrm{trace}\left[\left(\sum_{i=0}^{d_1+d_2-1}N_i^TN_i\right)\cdot\Sigma_a\right] \tag{12}$$

where $N_i(i=0,\ \ldots,\ d_1+d_2-1)$ are defined as the coefficient matrices of the matrix polynomial $[(Q_{11}+S_1q^{-d_1})\ (Q_{12}+S_2q^{-d_1})]$, and Σ_a is the variance-covariance matrix of the white noise vector $[a_1(k)\ a_2(k)]^T$.

Proof: See Appendix A.

Remark 1. The polynomials $Q_{11}(q^{-1})$ and $Q_{12}(q^{-1})$ are feedback-invariant and depend only on the disturbance characteristics affecting the primary loop. The polynomials $S_1(q^{-1})$ and $S_2(q^{-1})$ are also feedback-invariant, but they also depend on the primary process model, as well as the disturbance characteristics associated with the secondary loop.

Remark 2. The primary output $C_1(k)$ under the minimum variance cascade control is a moving average process of order d_1+d_2-1. Thus, the autocorrelation function of the primary output beyond lag d_1+d_2-1 will vanish. This result is similar to that of the single-loop case, and it can be used as a convenient tool for checking if the condition of minimum variance cascade control is being achieved.

Remark 3. The traditional method of a designing cascade control system (sequential approach) does not result in the same controllers given in Eqs. 3 and 4. This is because the minimum variance cascade controllers given in Theorem 1 minimize only the variance of the primary output without explicitly considering the variance of the secondary output, whereas, in the sequential approach, the secondary controller is designed first to achieve minimum variance in the secondary output variable. Indeed, the secondary output under the minimum variance cascade control can be derived as

$$C_2(k)=\left[Q_{21}-\frac{1}{G_1^*}(T_1+R_{11})q^{-d_2}\right]a_1(k)+\left[Q_{22}-\frac{1}{G_1^*}(T_2+R_{12})q^{-d_2}\right]a_2(k)$$

which shows that the local disturbance in the secondary loop is not regulated in a minimum variance sense.

An Example

Consider the following cascade control system

$$C_1(k)=\frac{1}{1-0.8q^{-1}}C_2(k-4)+\frac{1}{1-q^{-1}}a_1(k)$$

$$C_2(k)=\frac{1}{1-0.4q^{-1}}u_2(k-1)+\frac{1}{1-0.5q^{-1}}a_2(k)$$

where $\Sigma_a=I$. Solving the identities in Eqs. 5 through 10, we get

$$Q_{11}=1+q^{-1}+q^{-2}+q^{-3}+q^{-4}$$

$$Q_{22}=1$$

$$S_2=1$$

$$Q_{12}=Q_{21}=S_1=0$$

and

$$R_{11}=\frac{1}{1-q^{-1}},\ R_{22}=\frac{0.5}{1-0.5q^{-1}},\ T_2=\frac{0.8}{1-0.8q^{-1}}$$

$$R_{12}=R_{21}=T_1=0$$

The minimum variance cascade control law thus becomes

- *Primary Controller*

$$G_{c1}=\frac{G_1^*(Q_{22}R_{21}-Q_{21}R_{22})+(R_{11}+T_1)G_{L22}-(R_{12}+T_2)G_{L21}}{(Q_{11}+S_1q^{-d_1})(R_{12}+T_2+G_1^*R_{22})-(Q_{12}+S_2q^{-d_1})(R_{11}+T_1+G_1^*R_{21})}=\frac{1-0.8q^{-1}}{1.3-0.4q^{-1}-q^{-4}}$$

- *Secondary Controller*

$$G_{c2}=\frac{(Q_{11}+S_1q^{-d_1})(R_{12}+T_2+G_1^*R_{22})-(Q_{12}+S_2q^{-d_1})(R_{11}+T_1+G_1^*R_{21})}{G_2^*\left[G_{L11}S_2-G_{L12}S_1+(R_{11}Q_{12}-R_{12}Q_{11})q^{-d_2}\right]}=\frac{(1-0.4q^{-1})(1.3-0.4q^{-1}-q^{-4})}{(1-0.5q^{-1})(1-0.8q^{-1})}$$

The primary controller G_{c1} in this case has an unstable pole at $q = 1.024$, and this unstable pole is cancelled by the secondary controller G_{c2}. The output of the primary loop under minimum variance cascade control is

$$C_1(k) = \left(Q_{11} + S_1 q^{-d_1}\right) a_1(k) + \left(Q_{12} + S_2 q^{-d_1}\right) a_2(k)$$
$$= \left(1 + q^{-1} + q^{-2} + q^{-3} + q^{-4}\right) a_1(k) + q^{-4} a_2(k)$$

Therefore, the minimum variance is

$$\left(\sigma_{C_1}^2\right)_{MV} = \text{var}\left[\left(1+q^{-1}+q^{-2}+q^{-3}+q^{-4}\right) a_1(k) + q^{-4} a_2(k)\right]$$
$$= 5\sigma_{a_1}^2 + \sigma_{a_2}^2 = 6$$

Estimation of minimum achievable variance in cascade control system

In this section, we show that the minimum achievable output variance in the cascade control loop can be estimated using routine operating data and the knowledge of the process time delays without perturbing the process.

From Theorem 1 of the previous section, the primary output $C_1(k)$ under minimum variance cascade control is the following moving average process of order $d_1 + d_2 - 1$

$$C_1(k) = \left[Q_{11}(q^{-1}) + S_1(q^{-1}) q^{-d_1}\right] a_1(k) + \left[Q_{12}(q^{-1}) + S_2(q^{-1}) q^{-d_1}\right] a_2(k)$$

Since the polynomials $Q_{11}(q^{-1})$, $Q_{12}(q^{-1})$, $S_1(q^{-1})$, and $S_2(q^{-1})$ are all feedback-invariant, the expression for the primary output under minimum variance cascade control can be estimated from the first $d_1 + d_2 - 1$ moving average coefficients of the closed-loop transfer functions that relate $a_1(k)$ to $C_1(k)$ and $a_2(k)$ to $C_1(k)$. No joint identification of the process dynamics and the disturbance model is required. The closed-loop transfer functions in this case can be obtained from the first row of the transfer function matrix estimated via multivariate time-series modeling of $[C_1(k)\ C_2(k)]$. For the multivariate time-series modeling of $[C_1(k)\ C_2(k)]$, an autoregressive (AR) model formulation can be used efficiently with its computational speed and the capability of being recursively calculated (Desborough and Harris, 1992). In this case, the innovations sequences $\{a_1(k)\}$ and $\{a_2(k)\}$ are estimated as the residual vectors of multivariate multiple linear regression analyses. The sample variance-covariance matrix of these residual vectors can thus provide an estimate of the variance and the covariance elements of the innovations sequences. The closed-loop impulse response coefficients can then be determined via a simple correlation analysis between the output variables and the estimated innovations sequences, or by solving a suitable Diophantine identity concerning the estmated parameter matrix polynominal. Thus, minimum variance performance is estimated by

$$\left(\hat{\sigma}_{C_1}^2\right)_{MV} = \text{var}\left[\left\{\hat{Q}_{11}(q^{-1}) + \hat{S}_1(q^{-1}) q^{-d_1}\right\} a_1(k) + \left\{\hat{Q}_{12}(q^{-1}) + \hat{S}_2(q^{-1}) q^{-d_1}\right\} a_2(k)\right]$$
$$= \text{trace}\left[\left(\sum_{i=0}^{d_1+d_2-1} \hat{N}_i^T \hat{N}_i\right) \cdot \hat{\Sigma}_a\right] \tag{13}$$

where the symbol ^ is used to denote estimated values. In Eq. 13, $\hat{N}_i (i = 0, \ldots, d_1 + d_2 - 1)$ are defined as the coefficient matrices of the estimated matrix polynomial $[(\hat{Q}_{11} + \hat{S}_1 q^{-d_1})\ (\hat{Q}_{12} + \hat{S}_2 q^{-d_1})]$, and $\hat{\Sigma}_a$ is the estimated variance-covariance matrix of the white noise vector $[a_1(k)\ a_2(k)]^T$. It should be noted here that, in the performance assessment of control loops, the actual output variance $\sigma_{C_1}^2$ should be replaced by the output mean-square error $E(C_1^2)$ if an offset exists in the output variable.

When the net effect of disturbances driven by white noise sequence $\{a_2(k)\}$ is negligibly small compared to the one driven by white noise sequence $\{a_1(k)\}$, the minimum achievable variance in a cascade control loop can be approximated as

$$\left(\hat{\sigma}_{C_1}^2\right)_{MV} \cong \text{var}\left[\left\{\hat{Q}_{11}(q^{-1}) + \hat{S}_1(q^{-1}) q^{-d_1}\right\} a_1(k)\right]$$

In this case, the minimum achievable variance above can also be obtained from a univariate time-series modeling of the primary output $C_1(k)$. On the other hand, when there is an appreciable variance contribution due to the disturbances driven by white noise sequence $\{a_2(k)\}$, univariate time-series modeling does not give a proper performance measure in the cascade control system. In such cases, a multivariate time-series modeling of $[C_1(k)\ C_2(k)]$ should be employed to obtain the minimum achievable output variance given in Eq. 13.

When there are measured disturbances that affect the primary output, an ANOVA analysis can be carried out to estimate variance contributions from each measured and unmeasured disturbances to the total variance in the primary output. In this case the procedure developed by Desborough and Harris (1993) can be directly applied to the primary output for an ANOVA analysis, except the point that the unmeasured disturbance in the primary loop estimated from a lagged regression has to be used together with the secondary output for multivariate time series analysis to obtain minimum achievable variance from the feedback controller.

Admissible Minimum Variance Cascade Control of Processes with Unstable Zeros

Derivation of admissible minimum variance cascade control

When processes have unstable zeros, the minimum variance cascade control law given in Eqs. 3–4 can result in an input signal that is infinitely large. For a controller to be physically realizable, it is required that the controller is stable as well as proper. Such an admissible control law is derived in this section when the primary process $G_1(q^{-1})$ has noninvertible zeros.

The following theorem is an extension of the methodology used in single-loop case (Åström and Wittenmark, 1990) to the cascade control loop for the derivation of admissible minimum variance control law. For brevity, the backward shift operator q^{-1} is omitted in this section, that is, $G_1(q^{-1})$ will be denoted simply as G_1.

Theorem 2

Consider the following cascade control system where the primary process G_1 has unstable zeros, while the secondary process G_2 has a stable inverse

$$C_1(k) = G_1 C_2(k) + G_{L11} a_1(k) + G_{L12} a_2(k)$$
$$= G_1^+ G_1^- q^{-d_1} C_2(k) + \left(Q_{11} + R_{11} q^{-d_1-d_2}\right) a_1(k)$$
$$+ \left(Q_{12} + R_{12} q^{-d_1-d_2}\right) a_2(k) \quad (14)$$

$$C_2(k) = G_2 u_2(k) + G_{L21} a_1(k) + G_{L22} a_2(k)$$
$$= G_2^* q^{-d_2} u_2(k) + \left(Q_{21} + R_{21} q^{-d_2}\right) a_1(k)$$
$$+ \left(Q_{22} + R_{22} q^{-d_2}\right) a_2(k) \quad (15)$$

where G_1^+ is a monic polynomial that contains all the noninvertible zeros of G_1, and $G_1^- q^{-d_1}$ is the remaining term of G_1.

(i) The admissible minimum variance cascade control law is then given by

- Primary Controller

$$G_{c1} = \frac{\tilde{G}_1(Q_{22}R_{21} - Q_{21}R_{22}) + \left(R_{11}^{mp} + \tilde{T}_1\right)G_{L22} - \left(R_{12}^{mp} + \tilde{T}_2\right)G_{L21}}{(G_{L11} + G_1 G_{L21})\left(R_{12}^{mp} + \tilde{T}_2 + \tilde{G}_1 R_{22}\right) - (G_{L12} + G_1 G_{L22})\left(R_{11}^{mp} + \tilde{T}_1 + \tilde{G}_1 R_{21}\right)} \quad (16)$$

- Secondary Controller

$$G_{c2} = \frac{(G_{L11} + G_1 G_{L21})\left(R_{12}^{mp} + \tilde{T}_2 + \tilde{G}_1 R_{22}\right) - (G_{L12} + G_1 G_{L22})\left(R_{11}^{mp} + \tilde{T}_1 + \tilde{G}_1 R_{21}\right)}{G_2^*\left[G_{L11}\left(\tilde{S}_2 - R_{12}^{mp} q^{-d_2}\right) - G_{L12}\left(\tilde{S}_1 - R_{11}^{mp} q^{-d_2}\right)\right]} \quad (17)$$

where $\tilde{G}_1 \equiv q^{-n} G_1^+(q) G_1^-(q^{-1})$, $n = \deg(G_1^+)$; $\tilde{S}_1$ and $\tilde{S}_2$ are polynomials of order $d_2 - 1$, and R_{11}^{mp}, R_{11}^{nmp}, R_{12}^{mp}, R_{12}^{nmp}, $\tilde{T}_1$, and $\tilde{T}_2$ are proper transfer functions that satisfy the following identities

$$FR_{11} = R_{11}^{mp} + R_{11}^{nmp} \quad (18)$$

$$FR_{12} = R_{12}^{mp} + R_{12}^{nmp} \quad (19)$$

$$\tilde{G}_1 Q_{21} = \tilde{S}_1 + \tilde{T}_1 q^{-d_2} \quad (20)$$

$$\tilde{G}_1 Q_{22} = \tilde{S}_2 + \tilde{T}_2 q^{-d_2} \quad (21)$$

where R_{11}^{nmp}, R_{12}^{nmp} are terms that contain all the noninvertible zeros of $G_1^+(q^{-1})$ as its poles after the partial fraction expansion of FR_{11} and FR_{12}, respectively, where $F \equiv q^{-n} G_1^+(q)/G_1^+(q^{-1})$. R_{11}^{mp}, R_{12}^{mp} are the remaining terms after the partial fraction expansions.

(ii) The primary output $C_1(k)$ under the admissible minimum variance cascade control above is given by

$$C_1(k) = \left(Q_{11} + \frac{\tilde{S}_1}{F} q^{-d_1} + \frac{R_{11}^{nmp} q^{-d_1-d_2}}{F}\right) a_1(k)$$
$$+ \left(Q_{12} + \frac{\tilde{S}_2}{F} q^{-d_1} + \frac{R_{12}^{nmp} q^{-d_1-d_2}}{F}\right) a_2(k) \quad (22)$$

Proof. See Appendix B.

Remark 1. The terms Q_{11}, Q_{12}, $\tilde{S}_1/F$, $\tilde{S}_2/F$, R_{11}^{nmp}/F, and R_{12}^{nmp}/F in Eq. 22 are all feedback-invariant and they only depend on process dynamics and/or disturbance characteristics.

Remark 2. When there is no unstable zero in the primary process, it is evident that $F = 1$, $\tilde{G}_1 = G_1^*$, $R_{11}^{mp} = R_{11}$, $R_{12}^{mp} = R_{12}$, $R_{11}^{nmp} = R_{12}^{nmp} = 0$, $\tilde{S}_1 = S_1$, $\tilde{S}_2 = S_2$, $\tilde{T}_1 = T_1$, and $\tilde{T}_2 = T_2$. In this case, it can be shown that the admissible minimum variance control law in Eqs. 16–17 reduces to that for minimum-phase process given in Eqs. 3–4.

Estimation of minimum achievable variance

In this section, a procedure is presented for the estimation of achievable minimum variance in the cascade control loop when the primary process has unstable zeros. The estimation of achievable minimum variance in this case, however, requires knowledge on the location of unstable zeros in addition to knowledge of the process time delays.

To estimate the expression for the primary output $C_1(k)$ under the admissible minimum variance cascade control given in Eq. 22, we need first to obtain a closed-loop transfer function from $[a_1(k)\ a_2(k)]$ to $C_1(k)$. For this, the same procedure as described earlier can be used to obtain the closed-loop transfer function. Denote the closed-loop transfer function obtained in this way as G_{cl}. Then, multiply G_{cl} by F and expand the resulting terms as

$$FG_{cl}\left[a_1(k)\ a_2(k)\right]^T = \left(Q_{1,cl} + R_{1,cl} q^{-d_1-d_2}\right) a_1(k)$$
$$+ \left(Q_{2,cl} + R_{2,cl} q^{-d_1-d_2}\right) a_2(k)$$

where $Q_{1,cl}$ and $Q_{2,cl}$ are polynomials in q^{-1} of order $d_1 + d_2 - 1$, and $R_{1,cl}$ and $R_{2,cl}$ are proper functions. Since $Q_{1,cl}$ and $Q_{2,cl}$ are feedback-invariant, they provide estimates for the terms $FQ_{11} + \tilde{S}_1 q^{-d_1}$ and $FQ_{12} + \tilde{S}_2 q^{-d_1}$, respectively. Thus,

$$\hat{Q}_{11} + \frac{\hat{\tilde{S}}_1 q^{-d_1}}{F} = \frac{Q_{1,cl}}{F}$$
$$\hat{Q}_{12} + \frac{\hat{\tilde{S}}_2 q^{-d_1}}{F} = \frac{Q_{2,cl}}{F} \quad (23)$$

The terms R_{11}^{nmp} and R_{12}^{nmp} can be estimated from the transfer functions that has all unstable zeros of the primary process as their poles after the partial fraction expansion of $R_{1,cl}$ and $R_{2,cl}$, respectively (Huang et al., 1997).

Performance Assessment Examples

Example 1

In this example the performance assessment procedure developed in the second section is illustrated via a simulation example. The cascade control system considered is

$$C_1(k)=\frac{1}{1-0.9q^{-1}}C_2(k-2)+\frac{1}{1-0.8q^{-1}}a_1(k)+\frac{q^{-1}}{1-0.1q^{-1}}a_2(k)$$

$$C_2(k)=\frac{1}{1-0.5q^{-1}}u_2(k-1)+\frac{q^{-1}}{1-0.2q^{-1}}a_1(k)+\frac{1}{1-0.3q^{-1}}a_2(k)$$

where the white noise sequences $\{a_1(k)\}$ and $\{a_2(k)\}$ have the variance-convariance matrix $\Sigma_a=\begin{bmatrix}1 & 0.1\\ 0.1 & 1\end{bmatrix}$. To obtain the minimum achievable variance in the primary output, we first solve the following Diophantine identities (see Eqs. 5–10).

$$\frac{1}{1-0.8q^{-1}}=Q_{11}+R_{11}q^{-3}$$

$$\frac{q^{-1}}{1-0.1q^{-1}}=Q_{12}+R_{12}q^{-3}$$

$$\frac{q^{-1}}{1-0.2q^{-1}}=Q_{21}+R_{21}q^{-1}$$

$$\frac{1}{1-0.3q^{-1}}=Q_{22}+R_{22}q^{-1}$$

$$\frac{1}{1-0.9q^{-1}}Q_{21}=S_1+T_1q^{-1}$$

$$\frac{1}{1-0.9q^{-1}}Q_{22}=S_2+T_2q^{-1}$$

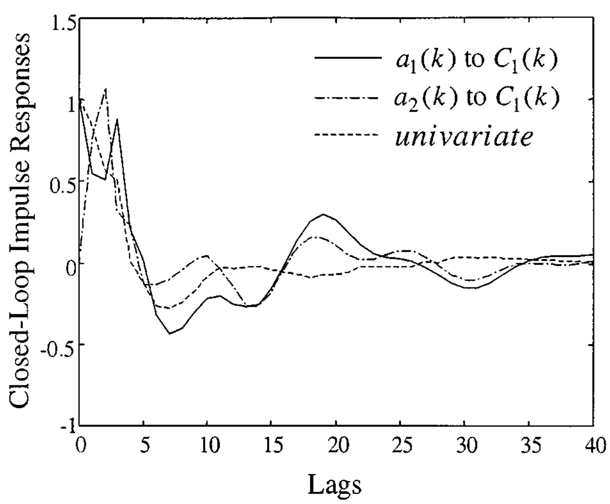

Figure 2. Closed-loop impulse response coefficients from simulated data.

where Q_{11} and Q_{12} are polynomials in q^{-1} of order 2, Q_{21}, Q_{22}, S_1 and S_2 are constants, and R_{ij} and T_i $(i,j=1,2)$ are proper transfer functions. The solution to these identities yields

$$Q_{11}=1+0.8q^{-1}+0.64q^{-2}$$

$$Q_{12}=q^{-1}+0.1q^{-2}$$

$$S_1=0$$

$$S_2=1$$

Thus, from Eq. 11, the primary output $C_1(k)$ under the minimum variance cascade control is given by

$$C_1(k)=\left[Q_{11}(q^{-1})+S_1(q^{-1})q^{-d_1}\right]a_1(k)+\left[Q_{12}(q^{-1})+S_2(q^{-1})q^{-d_1}\right]a_2(k)$$
$$=(1+0.8q^{-1}+0.64q^{-2})a_1(k)+(q^{-1}+1.1q^{-2})a_2(k)$$

The minimum achievable variance is then calculated as

$$\left(\sigma^2_{C_1}\right)_{MV}=\text{trace}\left\{\left(\begin{bmatrix}1\\0\end{bmatrix}[1\ \ 0]+\begin{bmatrix}0.8\\1\end{bmatrix}[0.8\ \ 1]+\begin{bmatrix}0.64\\1.1\end{bmatrix}[0.64\ \ 1.1]\right)\cdot\begin{bmatrix}1 & 0.1\\0.1 & 1\end{bmatrix}\right\}=4.560$$

Next, the achievable minimum variance is calculated from routine closed-loop data. A closed-loop simulation was performed using a proportional-integral (PI) controller for the primary loop and a proportional (P) controller for the secondary loop. The transfer functions for each controller were

- Primary Controller

$$G_{c1}=\frac{0.48-0.46q^{-1}}{1-q^{-1}}$$

- Secondary Controller

$$G_{c2}=0.7$$

2000 measurements for the primary and the secondary outputs were collected and an autoregressive multivariate time-series modeling was then performed using the collected data. In Figure 2, the closed-loop impulse responses relating $a_1(k)$ to $C_1(k)$ and $a_2(k)$ to $C_1(k)$ are plotted. The estimate of the minimum achievable output variance can then be calculated from Eq. 12

$$\left(\hat\sigma^2_{C_1}\right)_{MV}=\text{trace}\left\{\left(\begin{bmatrix}1\\0\end{bmatrix}[1\ \ 0]+\begin{bmatrix}0.547\\0.765\end{bmatrix}[0.547\ \ 0.765]+\begin{bmatrix}0.513\\1.064\end{bmatrix}[0.513\ \ 1.064]\right)\cdot\begin{bmatrix}1.315 & 0.326\\0.326 & 1.148\end{bmatrix}\right\}=4.653$$

The minimum achievable variance contribution due to the disturbances driven by the white noise sequence $\{a_2(k)\}$ consists of more than 42% of the minimum achievable variance.

Since the variance contribution due to the disturbances driven by the white noise sequence $\{a_2(k)\}$ is not negligible, the estimation of the minimum achievable variance using a univariate time-series modeling would lead to erroneous conclusions. In fact, a univariate time-series modeling of the primary output would yield the impulse response shown in Figure 2 as a dotted line, and, thus, the minimum achievable output variance could be obtained as

$$\left(\hat{\sigma}_{C_1}^2\right)_{MV} = (1+0.833^2+0.562^4)\cdot(2.745)$$
$$= 5.517$$

which is 19% larger than the actual minimum achievable variance.

Example 2

An industrial data set obtained from a level-to-flow cascade control system is analyzed in this example. The primary output in this case is the level of the overhead receiver in distillation column and the outlet flow rate of the overhead receiver is the secondary measurement point.

Figure 3 shows 3,000 measurements of the primary and the secondary outputs as deviation variables from its set point and steady-state value, respectively. For this process, it is known that $d_1 = 2$ and $d_2 = 1$. Figure 4 shows the closed-loop impulse responses obtained from an autoregressive multivariate time-series modeling of the primary and the secondary measurements. In this figure the closed-loop impulse responses relating $a_2(k)$ to $C_1(k)$ were small due to large capacity of the overhead receiver compared to the outlet flow rate. The estimated variance-covariance matrix of the innovations sequences was observed as

$$\hat{\Sigma}_a = \begin{bmatrix} 0.063 & 0.476 \\ 0.476 & 11.893 \end{bmatrix}.$$

Using the estimated impulse responses, the minimum achievable variance can be obtained as

$$\left(\hat{\sigma}_{C_1}^2\right)_{MV} = \mathrm{trace}\left\{\left(\begin{bmatrix}1\\0\end{bmatrix}[1\ \ 0] + \begin{bmatrix}0.734\\-0.001\end{bmatrix}[0.734\ \ -0.001] + \begin{bmatrix}0.496\\0.001\end{bmatrix}[0.496\ \ 0.001]\right)\cdot\begin{bmatrix}0.063 & 0.476\\0.476 & 11.893\end{bmatrix}\right\}$$
$$= 0.112$$

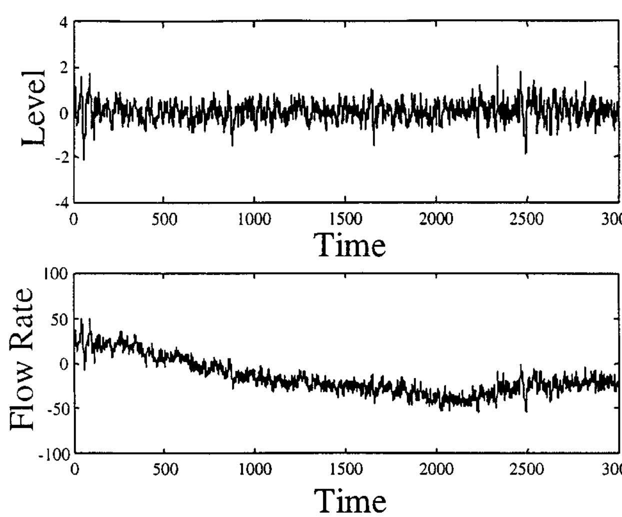

Figure 3. Industrial level-to-flow cascade control loop data.

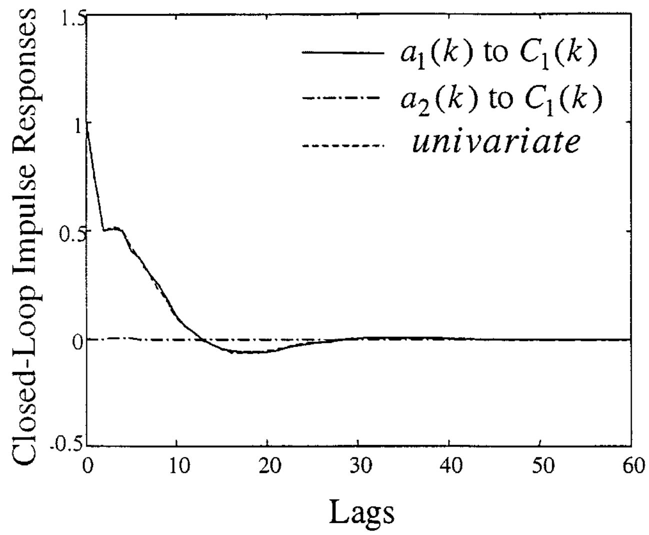

Figure 4. Closed-loop impulse response coefficients from industrial data.

It is interesting to note that there was negligible variance contribution due to the disturbances driven by the innovations sequence $\{a_2(k)\}$. Since there was negligible variance contribution due to the innovations sequence $\{a_2(k)\}$, a univariate time-series modeling can provide a similar results as with the multivariate time-series modeling. To verify this, a closed-loop impulse response was obtained from a univariate time-series modeling. Figure 4 compares this result to that obtained from multivariate time-series modeling, which shows essentially the same estimate of closed-loop impulse response with that from multivariate analysis. The univariate analysis in this case resulted in the value of 0.111 for an estimate of minimum achievable output variance.

This example illustrates that the minimum achievable output variance in the cascade control loop converges to that of the single loop feedback control as the disturbances driven by the innovations sequence $\{a_2(k)\}$ approach zero.

Conclusions

A simple method to estimate the minimum achievable variance in the cascade control loop has been developed for the performance monitoring of cascade control system. The minimum variance control law for the cascade control system was derived for cases where the primary process has (1) stable and (2) unstable inverses. The primary output under the minimum variance cascade control was found to be a moving average process of finite order, and the coefficients of this moving average process were observed to be independent of any causal feedback controller.

Based on the fact that the primary output under the minimum variance cascade control is feedback-invariant, an estimate of the minimum achievable variance has been obtained via multivariate time-series modeling of the primary and the secondary measurements collected under the normal operation. It is only necessary to have the knowledge on the time delays of the primary and the secondary processes. No plant tests were required for this method.

A methodology is also developed for the estimation of achievable minimum variance when the primary process has unstable zeros. The estimation of achievable minimum variance in this case, however, requires the knowledge of the location of unstable zeros in addition to the knowledge of the process time delays.

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Appendix A: Proof of Theorem 1

Referring to the block diagram of the cascade control system in Figure 1, the output of the primary loop can be written as follows

$$C_1(k) = \frac{(1+G_2G_{c2})\{G_{L11}a_1(k)+G_{L12}a_2(k)\}+G_1\{G_{L21}a_1(k)+G_{L22}a_2(k)\}}{1+G_2G_{c2}+G_1G_2G_{c1}G_{c2}} \tag{A1}$$

Define the following Diophantine identities

$$G_{L11} = Q_{11} + R_{11}q^{-d_1-d_2} \tag{A2}$$

$$G_{L12} = Q_{12} + R_{12}q^{-d_1-d_2} \tag{A3}$$

$$G_{L21} = Q_{21} + R_{21}q^{-d_2} \tag{A4}$$

$$G_{L22} = Q_{22} + R_{22}q^{-d_2} \tag{A5}$$

$$G_1^* Q_{21} = S_1 + T_1 q^{-d_2} \tag{A6}$$

$$G_1^* Q_{22} = S_2 + T_2 q^{-d_2} \tag{A7}$$

where Q_{11} and Q_{12} are polynomials in q^{-1} of order d_1+d_2-1, and Q_{21}, Q_{22}, S_1, and S_2 are polynomials in q^{-1} of order d_2-1. R_{ij} and T_i $(i,\ j=1,2)$ are proper transfer functions. Substituting the identities above into Eq. A1 and regrouping the terms into feedback-invariant and feedback-dependent terms, we get

$$C_1(k) = \left(Q_{11} + S_1 q^{-d_1}\right)a_1(k) + \left(Q_{12} + S_2 q^{-d_1}\right)a_2(k)$$

$$+ q^{-d_1-d_2}\underbrace{\left[\frac{(1+G_2G_{c2})R_{11} + G_1^* R_{21} + T_1 - Q_{11}G_1^* G_2^* G_{c1}G_{c2} - S_1 G_2^* G_{c2}(1+G_1G_{c1})}{1+G_2G_{c2} + G_1G_2G_{c1}G_{c2}}\right]}_{M_1} a_1(k)$$

$$+ q^{-d_1-d_2}\underbrace{\left[\frac{(1+G_2G_{c2})R_{12} + G_1^* R_{22} + T_2 - Q_{12}G_1^* G_2^* G_{c1}G_{c2} - S_2 G_2^* G_{c2}(1+G_1G_{c1})}{1+G_2G_{c2} + G_1G_2G_{c1}G_{c2}}\right]}_{M_2} a_2(k)$$

$$= \underbrace{\left(Q_{11} + S_1 q^{-d_1}\right)a_1(k) + \left(Q_{12} + S_2 q^{-d_1}\right)a_2(k)}_{\text{Feedback-Invariant}} + \underbrace{q^{-d_1-d_2}\left[M_1 a_1(k) + M_2 a_2(k)\right]}_{\text{Feedback-Dependent}} \quad \text{(A8)}$$

It is noted that M_1 and M_2 are both proper since the controller transfer function should be proper. Therefore, the feedback-invariant terms in Eq. A8 are independent to the feedback dependent terms. Hence, the variance of the primary output $\sigma_{C_1}^2$ satisfy the following inequality

$$\sigma_{C_1}^2 \geq \mathrm{var}\left\{\left(Q_{11} + S_1 q^{-d_1}\right)a_1(k) + \left(Q_{12} + S_2 q^{-d_1}\right)a_2(k)\right\}$$

$$= \mathrm{trace}\left[\left(\sum_{i=0}^{d_1+d_2-1} N_i^T N_i\right)\cdot \Sigma_a\right] \quad \text{(A9)}$$

where $N_i (i = 0, \ldots, d_1 + d_2 - 1)$ are defined as the coefficient matrices of the matrix polynomial $[(Q_{11} + S_1 q^{-d_1})\ (Q_{12} + S_2 q^{-d_1})]$, and Σ_a is the variance-covariance matrix of the white noise vector $[a_1(k)\ a_2(k)]^T$. Equality in Eq. A9 is obtained when the minimum variance cascade controller is used. The minimum variance cascade controller in this case can be obtained by solving the following coupled equation for the primary and the secondary controllers

$$M_1 = \frac{(1+G_2G_{c2})R_{11} + G_1^* R_{21} + T_1 - Q_{11}G_1^* G_2^* G_{c1}G_{c2} - S_1 G_2^* G_{c2}(1+G_1G_{c1})}{1+G_2G_{c2} + G_1G_2G_{c1}G_{c2}} = 0$$

$$M_2 = \frac{(1+G_2G_{c2})R_{12} + G_1^* R_{22} + T_2 - Q_{12}G_1^* G_2^* G_{c1}G_{c2} - S_2 G_2^* G_{c2}(1+G_1G_{c1})}{1+G_2G_{c2} + G_1G_2G_{c1}G_{c2}} = 0 \quad \text{(A10)}$$

The solution to this coupled equation can be obtained as the following proper functions:

- *Primary Controller*

$$G_{c1} = \frac{G_1^*(Q_{22}R_{21} - Q_{21}R_{22}) + (R_{11} + T_1)G_{L22} - (R_{12} + T_2)G_{L21}}{\left(Q_{11} + S_1 q^{-d_1}\right)(R_{12} + T_2 + G_1^* R_{22}) - \left(Q_{12} + S_2 q^{-d_1}\right)(R_{11} + T_1 + G_1^* R_{21})} \quad \text{(A11)}$$

- *Secondary Controller*

$$G_{c2} = \frac{\left(Q_{11} + S_1 q^{-d_1}\right)(R_{12} + T_2 + G_1^* R_{22}) - \left(Q_{12} + S_2 q^{-d_1}\right)(R_{11} + T_1 + G_1^* R_{21})}{G_2^*\left[G_{L11}S_2 - G_{L12}S_1 + (R_{11}Q_{12} - R_{12}Q_{11})q^{-d_2}\right]} \quad \text{(A12)}$$

Appendix B: Proof of Theorem 2

Consider the following cascade system

$$C_1(k) = G_1 C_2(k) + G_{L11}a_1(k) + G_{L12}a_2(k)$$

$$= G_1^+ G_1^- q^{-d_1} C_2(k) + \left(Q_{11} + R_{11} q^{-d_1-d_2}\right)a_1(k)$$

$$+ \left(Q_{12} + R_{12} q^{-d_1-d_2}\right)a_2(k) \quad \text{(B1)}$$

$$C_2(k) = G_2 u_2(k) + G_{L21}a_1(k) + G_{L22}a_2(k)$$

$$= G_2^* q^{-d_2} u_2(k) + \left(Q_{21} + R_{21} q^{-d_2}\right)a_1(k)$$

$$+ \left(Q_{22} + R_{22} q^{-d_2}\right)a_2(k) \quad \text{(B2)}$$

where $G_1 \equiv G_1^+ G_1^- q^{-d_1}$ has noninvertible zeros at the zeros of G_1^+, where G_1^+ is assumed to be a monic polynomial. Multiplying $F \equiv q^{-n} G_1^+(q)/G_1^+(q^{-1})$, $n = \deg\ (G_1^+)$ on both sides of Eq. B1, and substituting the expression for $C_2(k)$ in

Eq. B2 into Eq. B1 yields

$$
\begin{aligned}
FC_1(k) &= \left(FQ_{11} + \tilde{G}_1 Q_{21} q^{-d_1}\right) a_1(k) \\
&\quad + \left(FQ_{12} + \tilde{G}_1 Q_{22} q^{-d_1}\right) a_2(k) + q^{-d_1-d_2}\left[\tilde{G}_1 G_2^* u_2(k)\right. \\
&\quad \left. + \left(\tilde{G}_1 R_{21} + FR_{11}\right) a_1(k) + \left(\tilde{G}_1 R_{22} + FR_{12}\right) a_2(k)\right] \quad \text{(B3)}
\end{aligned}
$$

where $\tilde{G}_1 \equiv q^{-n} G_1^+(q) G_1^-(q^{-1})$. Since $F(q^{-1})F(q) = 1$, the signals $C_1(k)$ and $FC_1(k)$ have the same spectrum and thus the same variance.

Next, define the following identities

$$FR_{11} = R_{11}^{mp} + R_{11}^{nmp} \quad \text{(B4)}$$

$$FR_{12} = R_{12}^{mp} + R_{12}^{nmp} \quad \text{(B5)}$$

$$\tilde{G}_1 Q_{21} = \tilde{S}_1 + \tilde{T}_1 q^{-d_2} \quad \text{(B6)}$$

$$\tilde{G}_1 Q_{22} = \tilde{S}_2 + \tilde{T}_2 q^{-d_2} \quad \text{(B7)}$$

where R_{11}^{nmp}, R_{12}^{nmp} are terms that contain all the noninvertible zeros of $G_1^+(q^{-1})$ as their poles after the partial fraction expansion of FR_{11} and FR_{12}, respectively; R_{11}^{mp}, R_{12}^{mp} are remaining terms after the partial fraction expansions. Substituting the identities in Eqs. B4–B7 into Eq. B3 gives

$$
\begin{aligned}
FC_1(k) &= \underbrace{\left(FQ_{11} + \tilde{S}_1 q^{-d_1} + R_{11}^{nmp} q^{-d_1-d_2}\right) a_1(k) + \left(FQ_{12} + \tilde{S}_2 q^{-d_1} + R_{12}^{nmp} q^{-d_1-d_2}\right) a_2(k)}_{\equiv V_1} \\
&\quad + \underbrace{q^{-d_1-d_2}\left[\tilde{G}_1 G_2^* u_2(k) + \left(\tilde{G}_1 R_{21} + R_{11}^{mp} + \tilde{T}_1\right) a_1(k) + \left(\tilde{G}_1 R_{22} + R_{12}^{mp} + \tilde{T}_2\right) a_2(k)\right]}_{\equiv V_2} \quad \text{(B8)}
\end{aligned}
$$

According to Wiener (1949), the causal unstable operators $R_{11}^{nmp}(q^{-1})$, $R_{12}^{nmp}(q^{-1})$ can be interpreted as noncausal stable operators as follows. Without loss of generality, consider the case when $R^{nmp}(q^{-1}) = 1/(1 + aq^{-1})$ where $|a| > 1$. Since the shift operator has the norm $|q| = 1$, the following series expansion of $R^{nmp}(q^{-1})$ converges

$$
\begin{aligned}
R^{nmp}(q^{-1}) &= \frac{1}{1 + aq^{-1}} = \frac{1}{a} \frac{q}{\left(1 + \dfrac{q}{a}\right)} \\
&= \frac{q}{a}\left[1 - \frac{1}{a} q + \frac{1}{a^2} q^2 - \cdots\right]
\end{aligned}
$$

Therefore, the term $R^{nmp}(q^{-1})$ can be interpreted as a noncausal stable operator expanded in terms of the shift operator q. With this interpretation, the terms V_1 and V_2 in Eq. B8 are independent. Therefore

$$
\begin{aligned}
\sigma_{C_1}^2 = \sigma_{FC_1}^2 &\geq \mathrm{var}[V_1] \\
&= \mathrm{var}\left[\left(Q_{11} + \frac{\tilde{S}_1}{F} q^{-d_1} + \frac{R_{11}^{nmp} q^{-d_1-d_2}}{F}\right) a_1(k)\right. \\
&\quad \left. + \left(Q_{12} + \frac{\tilde{S}_2}{F} q^{-d_1} + \frac{R_{12}^{nmp} q^{-d_1-d_2}}{F}\right) a_2(k)\right] \quad \text{(B9)}
\end{aligned}
$$

where the equality in Eq. B9 holds when the control law is described by

$$
\begin{aligned}
u_2(k) &= -\frac{1}{\tilde{G}_1 G_2^*}\left[\left(\tilde{G}_1 R_{21} + R_{11}^{mp} + \tilde{T}_1\right) a_1(k)\right. \\
&\quad \left. + \left(\tilde{G}_1 R_{22} + R_{12}^{mp} + \tilde{T}_2\right) a_2(k)\right] \quad \text{(B10)}
\end{aligned}
$$

The primary output $C_1(k)$ under this admissible minimum variance cascade control can be obtained by substituting the Eq. B10 into the Eq. B8

$$
\begin{aligned}
C_1(k) &= \left(Q_{11} + \frac{\tilde{S}_1}{F} q^{-d_1} + \frac{R_{11}^{nmp} q^{-d_1-d_2}}{F}\right) a_1(k) \\
&\quad + \left(Q_{12} + \frac{\tilde{S}_2}{F} q^{-d_1} + \frac{R_{12}^{nmp} q^{-d_1-d_2}}{F}\right) a_2(k) \quad \text{(B11)}
\end{aligned}
$$

Next, we derive the control laws for the primary and the secondary loops. From the block diagram in Figure 1, the control signal $u_2(k)$ can be expressed as

$$u_2(k) = -\frac{\left[G_{c1} G_{c2} G_{L11} + (1 + G_1 G_{c1}) G_{c2} G_{L21}\right] a_1(k) + \left[G_{c1} G_{c2} G_{L12} + (1 + G_1 G_{c1}) G_{c2} G_{L22}\right] a_2(k)}{1 + G_2 G_{c2} + G_1 G_2 G_{c1} G_{c2}} \quad \text{(B12)}$$

Comparison of Eq. B12 with Eq. B10 results in the following coupled equation for G_{c1} and G_{c2}

$$
\begin{aligned}
\frac{\tilde{G}_1 R_{21} + R_{11}^{mp} + \tilde{T}_1}{\tilde{G}_1 G_2^*} &= \frac{G_{c1} G_{c2} G_{L11} + (1 + G_1 G_{c1}) G_{c2} G_{L21}}{1 + G_2 G_{c2} + G_1 G_2 G_{c1} G_{c2}} \\
\frac{\tilde{G}_1 R_{22} + R_{12}^{mp} + \tilde{T}_2}{\tilde{G}_1 G_2^*} &= \frac{G_{c1} G_{c2} G_{L12} + (1 + G_1 G_{c1}) G_{c2} G_{L22}}{1 + G_2 G_{c2} + G_1 G_2 G_{c1} G_{c2}}
\end{aligned}
\quad \text{(B13)}
$$

The solution of this coupled equation is the following admissible minimum variance control law:

- *Primary Controller*

$$G_{c1} = \frac{\tilde{G}_1(Q_{22}R_{21} - Q_{21}R_{22}) + \left(R_{11}^{mp} + \tilde{T}_1\right)G_{L22} - \left(R_{12}^{mp} + \tilde{T}_2\right)G_{L21}}{(G_{L11} + G_1G_{L21})\left(R_{12}^{mp} + \tilde{T}_2 + \tilde{G}_1R_{22}\right) - (G_{L12} + G_1G_{L22})\left(R_{11}^{mp} + \tilde{T}_1 + \tilde{G}_1R_{21}\right)} \quad \text{(B14)}$$

- *Secondary Controller*

$$G_{c2} = \frac{(G_{L11} + G_1G_{L21})\left(R_{12}^{mp} + \tilde{T}_2 + \tilde{G}_1R_{22}\right) - (G_{L12} + G_1G_{L22})\left(R_{11}^{mp} + \tilde{T}_1 + \tilde{G}_1R_{21}\right)}{G_2^*\left[G_{L11}\left(\tilde{S}_2 - R_{12}^{mp}q^{-d_2}\right) - G_{L12}\left(\tilde{S}_1 - R_{11}^{mp}q^{-d_2}\right)\right]} \quad \text{(B15)}$$

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